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κ-compactness, extent and the Lindelöf number in LOTS

Research Article

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Abstract: We study the behaviour of $\aleph$-compactness, extent and Lindelöf number in lexicographic products of linearly ordered spaces. It is seen, in particular, that for the case that all spaces are bounded all these properties behave very well when taking lexicographic products. We also give characterizations of these notions for generalized ordered spaces.

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1. Introduction

A *linearly ordered topological space* (abbreviated LOTS) is a triple $(X, \lambda(\leqslant), \leqslant)$, where $(X, \leqslant)$ is a linearly ordered set (abbreviated LOS) and $\lambda(\leqslant)$ is the usual interval topology defined by $\leqslant$ (i.e., $\lambda(\leqslant)$ is the topology generated by $\{\,]a, \rightarrow[\; : a \in X\} \cup \{\,]\leftarrow, a[\; : a \in X\}$ as a subbase, where $]a, \rightarrow[\; = \{x \in X : a < x\}$ and $]\leftarrow, a[\; = \{x \in X : x < a\}$). A *generalized ordered space* (abbreviated GO-space) is a triple $(X, \tau, \leqslant)$, where $(X, \leqslant)$ is a linearly ordered set and τ is a topology on X such that $\lambda(\leqslant) \subseteq \tau$ and τ has a base consisting of order convex sets, where a subset A of X is called *order convex*, or simply *convex* if $x \in A$ for every x lying between two points of A [4, 6].

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It is well known that a topological space (X,τ) is a GO-space together with some ordering $\leqslant_X$ on X if and only if (X,τ) is a topological subspace of some LOTS $\big(Y,\lambda(\leqslant_Y),\leqslant_Y\big)$ with $\leqslant_X = \leqslant_Y\!\restriction_X$, where the symbol $\leqslant_Y\!\restriction_X$ is the restriction of the order $\leqslant_Y$ to X, so any GO-space has a linearly ordered extension. Note that a LOTS $\big(Y,\lambda(\leqslant_Y),\leqslant_Y\big)$ is called a *linearly ordered extension* of a GO-space $(X,\tau,\leqslant_X)$ if $X\subset Y$, $\tau=\lambda(\leqslant_Y)\!\restriction_X$ and $\leqslant_X = \leqslant_Y\!\restriction_X$ [5]. Any GO-space X has a linearly ordered extension Y such that X is closed in Y (see below).

For the sake of completeness we give the following definition: An ordered pair (A,B) of disjoint subsets of a LOS X is said to be a *gap* if

(i) $X = A\cup B$,

(ii) $a<b$ whenever $a\in A$ and $b\in B$,

(iii) A has no maximal element, and B has no minimal element.

If furthermore $A=\emptyset$ or $B=\emptyset$, then (A,B) is called an *endgap*. Note that if (A,B) is a gap or endgap of a GO-space $(X,\tau,\leqslant)$, then A and B are open subsets of X. An ordered pair (A,B) of *open* disjoint subsets of a GO-space $(X,\tau,\leqslant)$ is called a *pseudogap* if it satisfies (i), (ii),

(iv) $A\neq\emptyset$, $B\neq\emptyset$,

and, (iv)$_\mathrm{l}$ or (iv)$_\mathrm{r}$ stated by

(iv)$_\mathrm{l}$ A has no maximal element, and B has a minimal element,

(iv)$_\mathrm{r}$ A has a maximal element, and B has no minimal element.

Consider the following linearly ordered extension of an arbitrary GO-space $(X,\tau,\leqslant)$. Define X^* to be a subset of $X\times\mathbb{Z}$, where the order on $X\times\mathbb{Z}$ is the lexicographic order, by

$$X^* \;=\; (X\times\{0\})\cup\big\{(x,n):\,[x,\rightarrow[\;\in\tau-\lambda(\leqslant),\; n<0\big\}\cup\big\{(x,n):\,]\leftarrow,x]\in\tau-\lambda(\leqslant),\; n>0\big\}.$$

Let X^* be a LOTS by the order topology. Then it is easily seen that X is homeomorphic to the closed subspace $X\times\{0\}$ of X^*. Consequently, X^* is a linearly ordered extension of X [6].

If X is a LOTS then it has no pseudogaps. We denote the set of all gaps of a LOS X plus X itself by X^+ and define a linear order on X^+ in a natural way. Namely, if $c=(A,B)$ is a gap in X then we let $a<c$ for all $a\in A$ and $c<b$ for all $b\in B$. If X is a LOTS and we introduce the order topology on X^+ we obtain a compactification of X. More generally, if X is a GO-space, then X^{*+} is a compact space which contains X as a subspace. The closure of X in X^{*+} is a compactification of X, which is called the *Dedekind compactification* (or *Dedekind completion*, where the term completion is with respect to the order), which we also denote by X^+. Note that if X is a LOTS then $X^*=X$.

If X is a LOS and for some $x\in X$ we write $x=0$, then this would mean that X has a minimal element denoted by 0. Analogously, $x=1$ means that x is the maximal element of X denoted by 1.

2. Compactness and lexicographic products

In the rest of the paper we assume that $|X|\geq 2$ for any LOS X. Let $X=\mathbb{L}_{\alpha<\mu}X_\alpha$ be the lexicographic product of X_α, $\alpha<\mu$, where μ is an ordinal number >1 and X_α is a LOS for every $\alpha<\mu$. It is not difficult to see that $s=(s_\alpha)$ is the maximal element of X if and only if s_α is the maximal element of X_α for every $\alpha<\mu$. Analogously, $s=(s_\alpha)$ is the minimal element of X if and only if s_α is the minimal element of X_α for every $\alpha<\mu$. Let us begin with the following observation which describes the supremum of a non-empty subset of X.

(a) Suppose that X_α is complete for all $\alpha<\mu$, and let $\emptyset\neq A\subseteq X$. Define s_γ by induction as follows. If s_α is defined for all $\alpha<\gamma$, then

$$s_\gamma=\sup\big\{x\in X_\gamma:\ \text{there exists } a\in A \text{ such that } a_\alpha=s_\alpha \text{ for all } \alpha<\gamma,\ \text{ and } a_\gamma=x\big\},$$

here we use the convention $a=(a_\gamma)_{\gamma<\mu}$ and $\sup\emptyset=0$. Then $s=(s_\gamma)_{\gamma<\mu}=\sup A$.

(b) Without the completeness assumption on X_α, for $\emptyset \neq A \subseteq X$, define $s = (s_\gamma)_\gamma < \mu$ as in (a) above, but with $s_\gamma \in X_\gamma^+$. Then $t = (t_\gamma)_{\gamma<\mu} \in X$ is equal to $\sup A$ if and only if $s \in X$ and $t = s$, or there exists $\beta < \nu$, where $\nu = \min\{\gamma : s_\gamma \notin X_\gamma\}$, such that $s_\gamma = 1$ for $\beta < \gamma \leq \nu$, s_β has an immediate successor z in X_β, and

$$t_\gamma = \begin{cases} s_\gamma & \text{if} \quad \gamma < \beta, \\ z & \text{if} \quad \gamma = \beta, \\ 0 & \text{if} \quad \gamma > \beta. \end{cases}$$

Example 2.1.
In relation to the above, let us consider the following two examples:

(a) Let $X = \mathbb{L}_{n<\omega_0} X_n$, where $X_n = [0,1]$ for all $n < \omega_0$, and $A = \mathbb{L}_{n<\omega_0} A_n$, where $A_2 = [0,1/2[$ and $A_n = [0,1/2]$ for all $n \neq 2$. Then $s = (1/2, 1/2, 0, 0, \dots)$, since $s_n = \sup\emptyset$ for all $n > 2$.

(b) For $\mu = 2$, $X_0 = \{0,1\}$ and $X_1 = [0,1[$, let $A = \{(0,x) : x \in [0,1[\}$. Then $\sup A = (1,0)$ but $s = (0,1)$.

The following result is known (we include a short proof).

Theorem 2.2.
X is complete if and only if X_α is complete for all $\alpha < \mu$.

Proof. Sufficiency follows from the observation (a) above, where one notes that if a lattice L is bounded and any non-empty subset of L has a supremum, then L is complete. Conversely, suppose $\emptyset \neq Z \subseteq X_\beta$ for some $\beta < \mu$. Since X is complete it has a maximal element $t = (t_\alpha)$, so that t_α is the maximal element of X_α for every $\alpha < \mu$, as mentioned above. Let $A = \{(a_\alpha) : a_\alpha = t_\alpha \text{ for } \alpha \neq \beta,\ a_\beta \in Z\}$. It is then not difficult to see that if $s = \sup A$, then $s_\beta = \sup Z$. The argument for inf is analogous. □

It is known that if X_α is treated as a LOTS, then X_α is complete if and only if it is compact (see for example [1, X.7.12]). Consequently, we have the following result that was proved in [3, Theorem 4.2.1].

Theorem 2.3.
The LOTS $X = \mathbb{L}_{\alpha<\mu} X_\alpha$ is compact if and only if X_α is a compact LOTS for every $\alpha < \mu$.

3. κ-compactness

Let κ be some infinite cardinal number.

Definition 3.1.
A LOS X is said to be *κ-complete* if every nonempty subset in X of cardinality $\leq \kappa$ has a sup and inf. An $\aleph_0$-complete set is a σ-complete set. A LOS X is said to be *Dedekind κ-complete* if every nonempty bounded subset in X of cardinality $\leq \kappa$ has a sup and inf.

Definition 3.2.
A topological space X is said to be *κ-compact* (or *initially κ-compact* [7]) if every open cover of X of cardinality $\leq \kappa$ has a finite subcover. An $\aleph_0$-compact space is a countably compact space.

Now let X be a GO-space. A gap (or pseudogap) (A, B) is said to be a κ-gap (or κ-pseudogap) if there is a cofinal subset in A or coinitial subset in B of cardinality $\leq \kappa$. It is understood that in the case of pseudogap the cofinal/coinitial subset is in the part of the pseudogap which does not have a maximal/minimal element.

Suppose that X is a LOS and (A, B) is a κ-gap and suppose further that there is a subset P of cardinality $\leq \kappa$ that is cofinal in A. Then it is not difficult to see that P does not have a supremum and, therefore, X is not κ-complete. Conversely, suppose that X is not κ-complete and P is a subset of cardinality $\leq \kappa$ that does not have a supremum in X. If P is not bounded above then $(P^{\leftarrow}, \emptyset)$ defines an κ-endgap in X, where by $P^{\leftarrow}$ we mean the set $\{x \in X : x \leq y$ for some $y \in P\}$. If it is bounded above, then by letting S be the set of all upper-bounds of P, we see that $(P^{\leftarrow}, S)$ defines a κ-gap in X. We thus obtain the following simple result.

Proposition 3.3.

A LOS X is κ-complete if and only if X does not have any κ-gaps.

We now prove the main result of this section.

Theorem 3.4.

A GO-space X is κ-compact if and only if X has neither κ-gaps nor κ-pseudogaps.

Proof. Suppose that X is a κ-compact GO-space and (A, B) is a κ-gap; suppose further that there is a subset P of cardinality $\leq \kappa$ that is cofinal in A. Then $\mathcal{U} = \{\,]\leftarrow, p[\, : p \in P\} \cup \{B\}$ is an open cover of X of cardinality $\leq \kappa$ that does not have a finite subcover. Analogously, if (A, B) is a κ-pseudogap, where for example A has a maximal element and B has no minimal element, and there exists a subset P of cardinality $\leq \kappa$ that is coinitial in B, then $\mathcal{U} = \{\,]p, \rightarrow[\, : p \in P\} \cup \{A\}$ is an open cover of X of cardinality $\leq \kappa$ that does not have a finite subcover.

Conversely, suppose X has neither κ-gaps nor κ-pseudogaps; then X is κ-complete by Proposition 3.3. Let $(F_\gamma)_{\gamma \in \Gamma}$ be a decreasing net of closed non-empty subsets of X and $|\Gamma| \leq \kappa$. Take $x_\gamma \in F_\gamma$ for every $\gamma \in \Gamma$. Let $y_\gamma = \sup_{\alpha \geq \gamma} x_\alpha$. Note that $\{x_\alpha : \alpha \geq \gamma\} \subseteq F_\gamma$. If $y_\gamma \notin F_\gamma$ then $[y_\gamma, \rightarrow[$ is open in X since F_γ is closed, and y_γ does not have an immediate predecessor. Consequently, $(]\leftarrow, y_\gamma[\,, [y_\gamma, \rightarrow[)$ defines a κ-pseudo-gap. It follows that $y_\gamma \in F_\gamma$ for every $\gamma \in \Gamma$. Consider $y = \limsup x_\gamma = \inf y_\gamma$. Again, if $y \notin F_\gamma$ then $]\leftarrow, y]$ is open in X since F_γ is closed, and y does not have an immediate successor. Consequently, $(]\leftarrow, y]\,,]y, \rightarrow[)$ defines a κ-pseudo-gap. It follows that $y \in F_\gamma$ for every $\gamma \in \Gamma$, so that $\bigcap_{\gamma \in \Gamma} F_\gamma \neq \emptyset$ and X is κ-compact. □

Corollary 3.5.

The following are equivalent for a LOTS X:

(a) *X is κ-compact;*

(b) *X is κ-complete;*

(c) *X does not have any κ-gaps.*

4. Lexicographic products of κ-compact spaces

As in Section 2, let $X = \mathbb{L}_{\alpha<\mu} X_\alpha$ be the lexicographic product of X_α, $\alpha < \mu$, $\mu > 1$, where X_α is a LOS for every $\alpha < \mu$. As in Section 3, let κ be some infinite cardinal number.

Theorem 4.1.

Let X_α be bounded for every $\alpha < \mu$. Then X is κ-complete if and only if X_α is κ-complete for all $\alpha < \mu$.

Proof. Suppose X_α is κ-complete for every $\alpha < \mu$ and let $\emptyset \neq A \subseteq X$, where $|A| \leq \kappa$. Define s_γ by induction as in observation (a) in Section 2, then $s = (s_\gamma)_{\gamma<\mu} = \sup A$. Likewise one can show the existence of $\inf A$, so that X is κ-complete.

Conversely, suppose $\emptyset \neq Z \subseteq X_\alpha$ for some $\alpha < \mu$, with $|Z| \leq \kappa$. Since X is bounded above it has a maximal element $t = (t_\alpha)$, so that t_α is the maximal element of X_α for every $\alpha < \mu$, as already mentioned in Section 2. Let

$A = \{(a_\beta) : a_\beta = t_\beta$ for $\beta \neq \alpha$, $a_\alpha \in Z\}$. It is then not difficult to see that if $s = \sup A$, then $s_\alpha = \sup Z$. The argument for inf is analogous. $\square$

Theorem 4.1 is not true if we do not require that X_α are bounded. Indeed, let us look at the following three examples. In the rest of the paper, for an element x of a LOS X, by x^+ we mean the immediate successor of x in X if it exists.

Example 4.2.
Let $\mu = 2$, $X_0 = [0, \omega_1)$ and $X_1 = [0, 1)$. Then X_1 is not σ-complete but X is σ-complete.

Proof. Let Z be a countable subset of X. We let $Z_\alpha = \{x \in X_\alpha :$ there exists $z = (z_\beta) \in Z$ with $z_\alpha = x\}$.

Case I: $s_0 = \sup Z_0 \notin Z_0$, then $z = (s_0, 0)$ is $\sup Z$.

Case II: $s_0 = \sup Z_0 \in Z_0$. Then let us look at the following two cases.

Subcase II (a): $s_1 = \sup Z_1$ exists in X_1. In this case $z = (s_0, s_1)$ is $\sup Z$.

Subcase II (b): $s_1 = \sup Z_1$ does not exist, i.e. Z_1 is cofinal in X_1. In this case $z = (s_0^+, 0)$ is $\sup Z$. $\square$

Example 4.3.
Let $\mu = 2$, $X_0 = [0, \omega_1)$ and $X_1 = (\omega_1, 0]$, i.e. $[0, \omega_1)$ with the inverse order. Then both X_0 and X_1 are σ-complete but X is not. Indeed, the set $A = \{(n, 0) : n < \omega_0\}$ is a countable subset of X that does not have a supremum in X.

Example 4.4.
Let $X = \mathbb{L}_{\alpha<\mu} X_\alpha$ be a LOTS where μ is a limit ordinal greater than κ. Let $X_\kappa =]\lambda, 0]$, where λ is any limit ordinal that is not cofinal with κ, and $X_\alpha = \{0, 1\}$ (with $0 < 1$) for $\alpha \neq \kappa$. Then X_α is compact (hence κ-compact) for all $\alpha \neq \kappa$ and X_κ is κ-compact. Now consider a κ-set (i.e. a nonempty set of cardinality $\leq \kappa$) $(z_\gamma)_{\gamma<\kappa} = ((z_{\gamma\alpha})_\alpha)_\gamma$ in X as follows:

$$z_{\gamma\alpha} = \begin{cases} 1 & \text{if} \quad \alpha < \gamma, \\ 0 & \text{if} \quad \alpha \geq \gamma. \end{cases}$$

If $t = (t_\alpha) = \sup(z_\gamma)$ in X exists, then it is evident that $t_\alpha = 1$ for $\alpha < \kappa$ and X_κ must have a minimal element, a contradiction. Consequently, X is not κ-compact.

Let us now analyse the situation of Theorem 4.1 if we do not require that X_α are bounded for every $\alpha < \mu$. In fact we have the following result.

Theorem 4.5.
Let $X = \mathbb{L}_{\alpha<\mu} X_\alpha$ be the lexicographic product of X_α, $\alpha < \mu$, $\mu > 1$, where X_α is a LOS for every $\alpha < \mu$. We let

$$\begin{aligned} M_0 &= \{i \in \mathbb{N} : \text{ there exists } \kappa\text{-set in } X_i \text{ without max but with sup}\} \quad \text{and} \quad m_0 = \inf M_0, \\ N_0 &= \{i \in \mathbb{N} : \text{ there exists } \kappa\text{-set in } X_i \text{ without sup}\} \quad \text{and} \quad n_0 = \inf N_0, \\ M_1 &= \{i \in \mathbb{N} : \text{ there exists } \kappa\text{-set in } X_i \text{ without min but with inf}\} \quad \text{and} \quad m_1 = \inf M_1, \\ N_1 &= \{i \in \mathbb{N} : \text{ there exists } \kappa\text{-set in } X_i \text{ without inf}\} \quad \text{and} \quad n_1 = \inf N_1, \end{aligned}$$

where $\inf \emptyset = \omega_0$. Then X is κ-complete if and only if the following conditions are satisfied:

(a) *X_0 is κ-complete;*

(b) *X_α is Dedekind κ-complete for every α;*

(c) *$0, 1 \in X_\alpha$ for $\alpha \geq \omega_0$;*

(d) (d_1) *$\alpha > m_0$ implies $0 \in X_\alpha$,*

(d_2) *$\alpha \geq n_0$ implies $0 \in X_\alpha$,* (d$_3$) *$n_0 < \omega_0$ implies every $x \in \mathbb{L}_{n<m} X_n$ has an immediate successor for any $m \in N_0$;*

(e) (e$_1$) *$\alpha > m_1$ implies $1 \in X_\alpha$,*

(e$_2$) *$\alpha \geq n_1$ implies $1 \in X_\alpha$,*

(e$_3$) *$n_1 < \omega_0$ implies that any $x \in \mathbb{L}_{n<m} X_n$ has an immediate predecessor for any $m \in N_1$.*

Proof. Suppose that X is κ-complete. Fix $x_\alpha < y_\alpha$ for every $\alpha < \mu$.

(a) To show that X_0 is κ-complete, take an arbitrary κ-set $(a_\gamma)_\gamma$ in X_0 and define a κ-set $(z_\gamma)_\gamma = ((z_{\gamma\alpha})_\alpha)_\gamma$ in X as follows:

$$z_{\gamma\alpha} = \begin{cases} a_\gamma & \text{if} \quad \alpha = 0, \\ y_\alpha & \text{if} \quad \alpha \neq 0. \end{cases}$$

Let $t = (t_\alpha) = \sup(z_\gamma)$ in X, which exists by assumption. It is then evident that $t_0 = \sup(a_\gamma)$ in X_0. Dually one can prove the result for inf and, therefore, X_0 is κ-complete.

(b) To show that X_{α_0} is Dedekind κ-complete, take an arbitrary κ-set $(a_\gamma)_\gamma$ that is bounded above in X_{α_0} and define a bounded above κ-set $(z_\gamma)_\gamma = ((z_{\gamma\alpha})_\alpha)_\gamma$ in X as follows:

$$z_{\gamma\alpha} = \begin{cases} a_\gamma & \text{if} \quad \alpha = \alpha_0, \\ y_\alpha & \text{if} \quad \alpha \neq \alpha_0. \end{cases}$$

Let $t = (t_\alpha) = \sup(z_\gamma)$ in X, which exists by assumption. It is then evident, since $(a_\gamma)_\gamma$ is bounded above, that $t_{\alpha_0} = \sup(a_\gamma)$ in X_{α_0}. Dually one can prove the result for inf and therefore, X_{α_0} is Dedekind κ-complete.

(c) To show that $0 \in X_\alpha$ for $\alpha \geq \omega_0$, define a sequence $(z_n)_n = ((z_{n\alpha})_\alpha)_n$ in X as follows:

$$z_{n\alpha} = \begin{cases} y_\alpha & \text{if} \quad \alpha < n, \\ x_\alpha & \text{if} \quad \alpha \geq n. \end{cases}$$

Let $t = (t_\alpha) = \sup(z_n)$ in X, which exists by assumption. It is then evident that $t_\alpha = y_\alpha$ for $\alpha < \omega_0$ and $t_\alpha = 0$ for $\alpha \geq \omega_0$. Consequently, $0 \in X_\alpha$ for $\alpha \geq \omega_0$. Dually one can prove that $1 \in X_\alpha$ for $\alpha \geq \omega_0$.

(d) If $m_0 = \omega_0$ then (d$_1$) follows from (c). Thus, suppose that $m_0 \in \mathbb{N}$. Let $(a_\gamma)_\gamma$ be a κ-set in X_{m_0} such that $s = \sup(a_\gamma)$ exists in X_{m_0} but s is not $\max(a_\gamma)$. Now define a κ-set $(z_\gamma)_\gamma = ((z_{\gamma\alpha})_\alpha)_\gamma$ in X as follows:

$$z_{\gamma\alpha} = \begin{cases} a_\gamma & \text{if} \quad \alpha = m_0, \\ y_\alpha & \text{if} \quad \alpha \neq m_0. \end{cases}$$

Let $t = (t_\alpha) = \sup(z_\gamma)$ in X, which exists by assumption. It is then evident that $t_{m_0} = s$ and $t_\alpha = 0 \in X_\alpha$ for every $\alpha > m_0$.

The case for n_0 is similar. Indeed, if $n_0 = \omega_0$ then (d$_2$) follows from (c). Now let $n_0 \in \mathbb{N}$ and $(a_\gamma)_\gamma$ be a κ-set in X_m, where $m \in N_0$, such that $\sup(a_\gamma)$ does not exist in X_m. Note that $(a_\gamma)_\gamma$ must be an unbounded set. One defines a κ-set $(z_\gamma)_\gamma = ((z_{\gamma\alpha})_\alpha)_\gamma$ in X as above and note that $(y_0, y_1, \ldots, y_{m-1})$ must have an immediate successor in $\mathbb{L}_{n<m} X_n$, say $(y_0, y_1, \ldots, y_{m-1})^+$. Let $t = (t_\alpha) = \sup(z_\gamma)$ in X, which exists by assumption. It is then evident that $(t_0, t_1, \ldots, t_{m-1}) = (y_0, y_1, \ldots, y_{m-1})^+$ and $t_\alpha = 0 \in X_\alpha$ for every $\alpha \geq m$.

(e) Analogously one can prove the dual statement to (d).

We now prove the converse. Suppose that (a)–(e) hold, then X_α is κ-complete for every $\alpha \geq \omega_0$ and $\alpha = 0$. Besides, the space X_n is Dedekind κ-complete for every $n \in \mathbb{N}$. Moreover, since X_α is a bounded κ-complete space for every $\alpha \geq \omega_0$ we have that $\mathbb{L}_{\omega_0 \leq \alpha < \mu} X_\alpha$ is also κ-complete by Theorem 4.1. We can, therefore, consider the product $\mathbb{L}_{0 \leq \alpha \leq \omega_0} X_\alpha$, where X_0 is κ-complete, X_{ω_0} is bounded κ-complete, and X_n, $0 < n < \omega_0$, are Dedekind κ-complete spaces.

Let A be a κ-set in X. Define s_γ by induction as follows. If s_α is defined for all $\alpha < \gamma$, then $s_\gamma = \sup A_\gamma$, where

$$A_\gamma = \{x \in X_\gamma : \text{ there exists } a \in A \text{ such that } a_\alpha = s_\alpha \text{ for all } \alpha < \gamma, \text{ and } a_\gamma = x\}.$$

and the supremum is taken in X_γ^+. If $A_\gamma \neq \emptyset$ for every $0 < \gamma < \omega_0$ then $s = (s_\gamma)_{\gamma \le \omega_0} = \sup A$ in X. Otherwise, there exists a first $0 < n < \omega_0$ such that $A_n = \emptyset$. In this case, either $s_{n-1} = \sup A_{n-1}$ is in $X_{n-1} \setminus A_{n-1}$ or in $X_{n-1}^+ \setminus X_{n-1}$. In the first case, we have $n - 1 \ge m_0$ and by (d$_1$) we obtain that $\sup A = (s_0, \dots, s_{n-1}, 0, \dots, 0)$ exists in X. In the second case, we have $n - 1 \ge n_0$ and by (d$_2$) and (d$_3$) we obtain that $\sup A = ((s_0, \dots, s_{n-2})^+, 0, \dots, 0)$ exists in X, where $(s_0, \dots, s_{n-2})^+$ is the immediate successor of $(s_0, \dots, s_{n-2})$. One can note that by hypothesis, $n_0 > 0$. □

When $\mu = 2$ we get the following two corollaries, the proofs of which directly follow from Theorem 4.5. For linearly ordered sets X and Y, we write $X \cdot Y$ instead of $\mathbb{L}_{\alpha<2} X_\alpha$, where $X_0 = X$ and $X_1 = Y$.

Corollary 4.6.

Let X, Y be LOS. Then $X \cdot Y$ is κ-complete if and only if the following conditions are satisfied:

(a) *X is κ-complete and Y is Dedekind κ-complete;*

(b) *if X has a κ-set without a maximum (resp. minimum) then $0 \in Y$ (resp. $1 \in Y$);*

(c) *if Y has a κ-set without supremum (resp. infimum) then $0 \in Y$ (resp. $1 \in Y$) and every $x \in X$ has an immediate successor (resp. immediate predecessor).*

Corollary 4.7.

If X, Y are LOS such that Y is bounded, then $X \cdot Y$ is κ-complete if and only if X and Y are κ-complete.

5. Extent of lexicographic products

Let us recall the following cardinal invariant.

Definition 5.1.

A subset $A \subseteq X$ of a topological space X is said to be *discrete* if the subspace topology of A is the discrete topology. The *extent* $e(X)$ of X is the least infinite cardinal number κ such that every closed, discrete subset in X has cardinality $\le \kappa$.

Remark 5.2.

Let us note that a subset A of a topological space X is closed, discrete if every point in X has a neighbourhood that contains at most one point of A. In other words, the collection $\{\{a\} : a \in A\}$ is a discrete collection of subsets of X.

We now turn to the study of the extent of a lexicographic product of LOTS. Let us first consider the bounded case.

Lemma 5.3.

Let X, Y be LOTS, where Y is bounded, and let D be a closed discrete subset of $X \cdot Y$. Furthermore, let

$$D_X = \{x \in X : \text{ there exists } y \in Y \text{ with } (x, y) \in D\}$$

be the projection of D on X. Then D_X is a closed discrete subset of X.

Proof. Let $x \in X$ and for $i = 0, 1$ choose neighbourhoods U_i of (x, i) such that $U_i \cap D \subseteq \{(x, i)\}$. We may assume that

$$\begin{array}{llll} U_0 = \big](x_0, 1), (x, y_0)\big[& \text{with} \quad x_0 < x, \quad y_0 > 0 & (\text{if } x > 0) & \text{or} \\ U_0 = \big[(x_0, 0), (x, y_0)\big[& \text{with} \quad x_0 = 0, \quad y_0 > 0 & (\text{if } x = 0) & \\ U_1 = \big](x, y_1), (x_1, 0)\big[& \text{with} \quad x < x_1, \quad y_1 < 1 & (\text{if } x < 1) & \text{or} \\ U_1 = \big](x, y_1), (x_1, 1)\big] & \text{with} \quad x_1 = 1, \quad y_1 < 1 & (\text{if } x = 1). & \end{array}$$

It is enough to show that $]x_0, x_1[\cap D_X \subseteq \{x\}$. Let $x' \in]x_0, x_1[\cap D_X$ and $y' \in Y$ with $(x', y') \in D$.

Suppose $x' < x$. Then $(x_0, 1) < (x', y') < (x, y_0)$, so that $(x', y') \in U_0 \cap D \subseteq \{(x, 0)\}$ and, consequently, $x' = x$ giving a contradiction. On the other hand, if $x' > x$ then $(x, y_1) < (x', y') < (x_1, 0)$. Thus $(x', y') \in U_1 \cap D \subseteq \{(x, 1)\}$ and consequently, $x' = x$ again giving a contradiction. □

An analogous result for the projection of D on Y is not true, for example, consider $X = [0, \omega_0[$, $Y = [0, \omega_0]$ and $D = \{(n, n) : 0 < n < \omega_0\}$. However, we have the following result.

Lemma 5.4.

Let X, Y be LOTS, and let D be a closed discrete subset of $X \cdot Y$. Then, for every $x \in X$,

$$D_x = \{y \in Y : (x, y) \in D\} \subseteq Y$$

is a closed discrete subset of Y.

Proof. Let $x \in X$. Take any $y \in Y$ and consider $(x, y) \in X \cdot Y$. There exists a convex neighbourhood of (x, y) in $X \cdot Y$ that contains at most one point of D. It is not difficult to see that this implies a convex neighbourhood of y in Y that contains at most one point of D_x. Consequently, D_x is a closed discrete subset of Y. □

Lemma 5.5.

Let X_k be LOTS for $k = 0, \dots, n$, where X_k is bounded for $k = 1, \dots, n$ and suppose that D_k is a closed, discrete subset of X_k for $k = 0, \dots, n$. Then $D_0 \times \cdots \times D_n$ is closed and discrete in $\mathbb{L}_{0 \le k \le n} X_k$.

Proof. It follows by induction that it is enough to prove the result for the product of two spaces X_0 and X_1, where X_1 is bounded. We need to show that every point $x = (x_0, x_1) \in X_0 \cdot X_1$ has a neighbourhood U satisfying $|(D_0 \times D_1) \cap U| \le 1$. Let U_1 be a convex neighbourhood of x_1 in X_1 such that $D_1 \cap U_1 \subseteq \{x_1\}$.

(a) $0 < x_1 < 1$ or $x = (1, 1)$. In this case, let $U = \{x_0\} \times U_1$.

(b) $x_1 = 1$ and $x_0 < 1$. Suppose $U_1 =]x_1', 1]$, where $x_1' < 1$. Choose $x_0'' > x_0$ such that $D_0 \cap [x_0, x_0''[\subseteq \{x_0\}$. Then $U =](x_0, x_1'), (x_0'', 0)[$ is a neighbourhood of (x_0, x_1) satisfying the required condition. Indeed, if $(a_0, a_1) \in (D_0 \times D_1) \cap U$, then $a_0 \in D_0$, $a_1 \in D_1$ and $(x_0, x_1') < (a_0, a_1) < (x_0'', 0)$. Consequently, $x_0 \le a_0 < x_0''$, so that $a_0 = x_0$. Furthermore, $(x_0, x_1') < (x_0, a_1)$ implies $x_1' < a_1$ so that $a_1 \in D_1 \cap U_1$ and consequently, $a_1 = x_1$.

The dual statements for $x = (0, 0)$ or $x_1 = 0$ and $x_0 > 0$ are proved analogously. □

We remark that an analogous statement for infinite products is not true due to Theorem 5.9.

Corollary 5.6.

If X, Y are LOTS such that Y is bounded, then $e(X \cdot Y) = \max\{e(X), e(Y)\}$.

Proof. The fact that $e(X \cdot Y) \ge \max\{e(X), e(Y)\}$ follows from Lemma 5.5 since a singleton is a closed discrete subset. Conversely, suppose D is a closed discrete subset of $X \cdot Y$ and let D_X be the projection of D on X, as in Lemma 5.3. Then $D = \bigcup_{x \in D_X} D_x$, where D_x is as in Lemma 5.4, so that $|D_x| \le e(Y)$. Thus $|D| \le |D_X| \cdot e(Y) \le e(X) \cdot e(Y) \le \max\{e(X), e(Y)\}$. □

Using induction one can easily see that if X_k are linearly ordered spaces for $0 \leq k \leq n$ such that $X_1, \ldots, X_n$ are bounded, then $e(\mathbb{L}_{0\leq k\leq n} X_k) = \max_{0\leq k\leq n} e(X_k)$. In view of the above results one asks whether $e(X) \leq \sup_{\alpha<\mu} e(X_\alpha)$, where $X = \mathbb{L}_{\alpha<\mu} X_\alpha$. Let us begin with the following lemma.

Lemma 5.7.

Let $X = \mathbb{L}_{\alpha<\mu} X_\alpha$, where X_α is a bounded LOTS for all $\alpha > 0$, and $D \subseteq X$ a closed, discrete subset. Suppose further we are given some limit ordinal $\gamma \leq \mu$ and $\bar{x} = (\bar{x}_\alpha)_{\alpha<\gamma}$. Then there exists $\nu < \gamma$ with the property that there exists at most one $y \in D$ such that $y_\alpha = \bar{x}_\alpha$ for all $\alpha < \sigma$ and $y_\sigma < \bar{x}_\sigma$ for some $\nu < \sigma < \gamma$.

Proof. One may assume that $\bar{x} \neq 0$ as otherwise result is obvious. Let $x = (x_\alpha) \in X$ be defined by

$$x_\alpha = \begin{cases} \bar{x}_\alpha & \text{if} \quad \alpha < \gamma, \\ 0 & \text{if} \quad \alpha \geq \gamma. \end{cases}$$

Note that $x = \bar{x}$ if $\gamma = \mu$. There exists $a < x < b$ such that $]a, b[\cap D$ contains at most one point. Since $a < x$, there exists $\nu < \gamma$ such that $a_\alpha = x_\alpha$ for all $\alpha < \nu$ and $a_\nu < x_\nu$. Now suppose that there exists $y, z \in D$ and $\nu < \sigma_1, \sigma_2 < \gamma$ with the property that $y_\alpha = \bar{x}_\alpha$ for all $\alpha < \sigma_1$ and $y_{\sigma_1} < \bar{x}_{\sigma_1}$, and $z_\alpha = \bar{x}_\alpha$ for all $\alpha < \sigma_2$ and $z_{\sigma_2} < \bar{x}_{\sigma_2}$. This implies that $a < y, z < x$ and therefore, $y = z$. □

Remark 5.8.

The dual statement of Lemma 5.7 also holds.

Theorem 5.9.

Let $X = \mathbb{L}_{\alpha<\mu} X_\alpha$, where X_α is a bounded LOTS for all $\alpha > 0$. Then $e(X) = \sup_{\alpha<\mu} e(X_\alpha)$.

Proof. To see that $e(X_\alpha) \leq e(X)$ for every $\alpha < \mu$, let us note that for any given $\beta < \mu$, we have

$$\mathbb{L}_{\alpha<\mu} X_\alpha = \mathbb{L}_{\alpha<\beta} X_\alpha \cdot X_\beta \cdot \mathbb{L}_{\beta<\alpha<\mu} X_\alpha,$$

and a singleton is a closed discrete subset. Consequently, if $s = (s_\alpha)$ and D is a closed discrete subset of X_β, then the subset $E = \{(x_\alpha) : x_\alpha = s_\alpha \text{ for } \alpha \neq \beta \text{ and } x_\beta \in D\}$ is closed and discrete in X.

We now show that $e(X) \leq \sup_{\alpha<\mu} e(X_\alpha)$. Thus suppose that $\sup_{\alpha<\mu} e(X_\alpha) = \kappa$ and that there exists a closed discrete subset $D \subseteq X$ with $|D| > \kappa$. We now define by induction a strictly increasing sequence $\gamma_0 < \gamma_1 < \gamma_2 < \ldots < \mu$ of ordinal numbers, $x_{\gamma_n} \in X_{\gamma_n}$ and $x_\alpha \in X_\alpha$ for $\gamma_{n-1} < \alpha < \gamma_n$, $n \geq 1$, such that

$$D_n = \{y \in D : y_\alpha = x_\alpha \text{ for all } \alpha \leq \gamma_n\}$$

has cardinality $> \kappa$, and there exists $y \in D_{n-1}$ with $y_{\gamma_n} \neq x_{\gamma_n}$ if $n \geq 1$. For the sake of convenience, let us denote by $\pi_\gamma(y) = y_\gamma$, the projection of X onto X_γ.

$[n = 0]$ Let $\gamma_0 = 0$. Since

$$D = \bigcup_{\xi\in\pi_0(D)} \{y \in D : \pi_0(y) = \xi\},$$

$|\pi_0(D)| \leq \kappa$, by Lemma 5.3, and $|D| > \kappa$ it follows that there exists $\eta \in X_0$ such that $D_0 = \{y \in D : \pi_0(y) = \eta\}$ has cardinality $> \kappa$. Let $x_{\gamma_0} = \eta$.

$[n-1 \to n]$ Since $|D_{n-1}| > \kappa$ and for every $y \in D_{n-1}$, $y_\alpha = x_\alpha$ for all $\alpha \leq \gamma_{n-1}$, we have $\gamma_n = \min\{\alpha : \text{there exists } y, z \in D_{n-1} \text{ with } y_\alpha \neq z_\alpha\} > \gamma_{n-1}$. Now

$$D_{n-1} = \bigcup_{\xi\in\pi_{\gamma_n}(D_{n-1})} \{y \in D_{n-1} : \pi_{\gamma_n}(y) = \xi\},$$

so that as above, there exists $\eta \in X_{\gamma_n}$ such that $D_n = \{y \in D_{n-1} : \pi_{\gamma_n}(y) = \eta\}$ has cardinality $> \kappa$. We now let $x_\alpha = \pi_\alpha(y)$, $\gamma_{n-1} < \alpha \leq \gamma_n$, for $y \in D_n$. One can note that this does not depend on the choice of y.

We thus have that $y_\alpha = x_\alpha$, $\alpha \leq \gamma_n$, for every $y \in D_n$, $D \supseteq D_n \searrow$ for all n, and finally, there exists $y \in D_{n-1}$ such that $y_{\gamma_n} \neq x_{\gamma_n}$, $n \geq 1$. If in Lemma 5.7, we let $\gamma = \sup \gamma_n$ and $\overline{x} = (x_\alpha)_{\alpha<\gamma}$, we obtain a contradiction. Indeed, according to Lemma 5.7 and its dual statement, there exists $\nu < \gamma$ with the property that there exists at most one $y \in D$ such that $y_\alpha = \overline{x}_\alpha$ for all $\alpha < \sigma$ and $y_\sigma < \overline{x}_\sigma$ for some $\nu < \sigma < \gamma$, and at most one $y \in D$ such that $y_\alpha = \overline{x}_\alpha$ for all $\alpha < \sigma'$ and $y_{\sigma'} > \overline{x}_{\sigma'}$ for some $\nu < \sigma' < \gamma$ and, therefore, the set $D' = \{(y_\alpha)_{\alpha<\gamma} : y \in D, (y_\alpha)_{\alpha\leq\nu} = (x_\alpha)_{\alpha\leq\nu}\}$ contains at most three elements. On the other hand, there exists $N \in \mathbb{N}$ such that $\nu < \gamma_N < \gamma_{N+1} < \ldots < \gamma$, and according to our construction, for every $n \geq N$, there exists $y \in D_n \subseteq D$, and therefore $y_\alpha = x_\alpha$, $\alpha \leq \gamma_n$, such that $y_{\gamma_{n+1}} \neq x_{\gamma_{n+1}}$, i.e. D' is infinite, a contradiction. $\square$

It is surprising to note that Theorem 5.9 does not depend on the cardinality of μ. Lemma 5.7 also enables us to obtain the following interesting result.

Proposition 5.10.

Let $X = \mathbb{L}_{\alpha<\mu} X_\alpha$, where X_α is a bounded LOTS for all $\alpha > 0$, and let $D_\alpha \neq \emptyset$ be a closed, discrete subset of X_α for all $\alpha < \mu$. Then $\prod D_\alpha$ is closed discrete in X if and only if $\{\alpha : |D_\alpha| > 1\}$ is finite.

Proof. Sufficiency follows from Lemma 5.5 and the already noted fact that singletons are closed and discrete.

To prove necessity, suppose $\gamma_1 < \gamma_2 < \ldots < \mu$ satisfy $|D_{\gamma_k}| > 1$ for $k = 1, 2, \ldots$ Suppose further that $\prod D_\alpha$ is closed, discrete in X. Let $\gamma = \sup \gamma_k$ and note that γ is a limit ordinal. Since $\prod D_\alpha = \prod_{\alpha<\gamma} D_\alpha \times \prod_{\gamma\leq\alpha<\mu} D_\alpha$, it follows from Lemma 5.3 that $\prod_{\alpha<\gamma} D_\alpha$ is closed discrete in $\mathbb{L}_{\alpha<\gamma} X_\alpha$. We are only left to apply Lemma 5.7, and its dual statement, to obtain a contradiction. Indeed, take any $x = (x_\alpha) \in D$. According to Lemma 5.7 and its dual statement, there exists $\nu < \gamma$ with the property that there exists at most two other elements in D whose first index in which they differ from x lies between ν and γ. On the other hand, there exists $N \in \mathbb{N}$ such that $\nu < \gamma_N < \gamma_{N+1} < \ldots < \gamma$, and since $|D_{\gamma_k}| > 1$ for all k, it follows that there are uncountably many such elements of D. $\square$

Let us now turn to the case when the LOTS X_α are not all bounded for $\alpha > 0$. We first consider LOTS that do not have endpoints, i.e. have neither 0 nor 1.

Theorem 5.11.

Suppose X, Y are LOTS such that $0, 1 \notin Y$, then

$$e(X \cdot Y) = \max\{|X|, e(Y)\} = |X| \,.\, e(Y).$$

Proof. Given $y \in Y$, the subset $X \times \{y\}$ is closed, discrete in $X \cdot Y$. Consequently, $e(X \cdot Y) \geq |X|$. Also, given $x \in X$ and a closed, discrete subset D in Y, we have $\{x\} \times D$ is closed, discrete in $X \cdot Y$ and therefore, $e(X \cdot Y) \geq e(Y)$. Thus, $e(X \cdot Y) \geq |X| \,.\, e(Y)$. To prove the reverse inequality, let D be a closed, discrete subset of $X \cdot Y$. For any $x \in X$, the subset $D_x = \{y \in Y : (x, y) \in D\}$ is closed, discrete in Y, and therefore $|D_x| \leq e(Y)$. Now $D = \bigcup_{x\in X} D_x$, so that $|D| \leq |X| \,.\, e(Y)$ and the required inequality follows. $\square$

Corollary 5.12.

Let $X = \mathbb{L}_{\alpha\leq\mu} X_\alpha$, where $0, 1 \notin X_\mu$. Then $e(X) = |\mathbb{L}_{\alpha<\mu} X_\alpha| \,.\, e(X_\mu)$.

Corollary 5.13.

Let $X = \mathbb{L}_{\alpha<\mu} X_\alpha$ and suppose that there exists $\gamma < \mu$ such that $0, 1 \notin X_\gamma$ and X_α are bounded for all $\gamma < \alpha < \mu$. Then

$$e(X) = e(\mathbb{L}_{\alpha\leq\gamma} X_\alpha) \vee \sup_{\gamma<\alpha<\mu} e(X_\alpha) = |\mathbb{L}_{\alpha<\gamma} X_\alpha| \,.\, \sup_{\gamma\leq\alpha<\mu} e(X_\alpha).$$

This corollary follows from applying Theorem 5.9 to $\mathbb{L}_{\alpha\leq\gamma} X_\alpha \cdot \mathbb{L}_{\gamma<\alpha<\mu} X_\alpha$ and then Corollary 5.12 to $\mathbb{L}_{\alpha\leq\gamma} X_\alpha$. Before going to our next theorem we need the following lemma.

Lemma 5.14.

Let D be a closed, discrete subset of $X = \mathbb{L}_{\alpha<\mu} X_\alpha$. Given any $x = (x_\alpha) \in D$, there exists $\alpha_0 < \mu$ such that $y = (y_\alpha) \in D$ and $x_\alpha = y_\alpha$ for all $\alpha \leq \alpha_0$ implies $x = y$.

Proof. Let $x \in D$, $0 < x < 1$ (the proof for $x = 0$ or $x = 1$ can be proved analogously). There exists $u, v \in X$ such that $u < x < v$ and $]u, v[\cap D = \{x\}$. Let $\alpha_1 = \min\{\alpha : u_\alpha \neq x_\alpha\}$, $\alpha_2 = \min\{\alpha : v_\alpha \neq x_\alpha\}$ and $\alpha_0 = \max\{\alpha_1, \alpha_2\}$. It is not difficult to see that if $y \in D$ and $x_\alpha = y_\alpha$ for all $\alpha \leq \alpha_0$, then $y \in D\cap]u, v[$ and therefore, $y = x$. □

Theorem 5.15.

Let $X = \mathbb{L}_{\alpha<\mu} X_\alpha$ and suppose that $\{\alpha : 0, 1 \notin X_\alpha\}$ is cofinal in μ, where μ is a limit ordinal. Then

$$e(X) = \sup_{\beta<\mu} \left| \prod_{\alpha\leq\beta} X_\alpha \right|. \tag{$*$}$$

Proof. Let us first note that given any $\beta < \mu$, $\mathbb{L}_{\beta<\alpha<\mu} X_\alpha$ does not have 0 and 1. So, $e(X) = |\mathbb{L}_{\alpha\leq\beta} X_\alpha| \,.\, e(\mathbb{L}_{\beta<\alpha<\mu} X_\alpha) \geq |\mathbb{L}_{\alpha\leq\beta} X_\alpha|$. The inequality $e(X) \geq \sup_{\beta<\mu} |\prod_{\alpha\leq\beta} X_\alpha|$ then follows. To get the inverse inequality, suppose D is a closed discrete subset of X and, for $\beta < \mu$, let

$$D_\beta = \bigl\{x \in D : y \in D,\, x_\alpha = y_\alpha \text{ for all } \alpha \leq \beta \Rightarrow x = y\bigr\}.$$

It follows from Lemma 5.14 that $D = \bigcup_{\gamma<\mu} D_\gamma$. Since $|D_\beta| \leq \left|\prod_{\alpha\leq\beta} X_\alpha\right| \leq \sup_{\gamma<\mu} \left|\prod_{\alpha\leq\gamma} X_\alpha\right|$, we have $|D| \leq |\mu| \,.\, \sup_{\gamma<\mu} \left|\prod_{\alpha\leq\gamma} X_\alpha\right|$. We are only left to show that $|\mu| \leq \sup_{\gamma<\mu} \left|\prod_{\alpha\leq\gamma} X_\alpha\right|$. Indeed, for $\beta < \mu$, $\sup_{\gamma<\mu} \left|\prod_{\alpha\leq\gamma} X_\alpha\right| \geq \left|\prod_{\alpha\leq\beta} X_\alpha\right| \geq |\beta|$, so that $\sup_{\gamma<\mu} \left|\prod_{\alpha\leq\gamma} X_\alpha\right| \geq |\mu|$ as required to show. □

We can now state a general result for products of LOTS which are either bounded or do not contain 0 and 1.

Theorem 5.16.

Consider the product $X = \mathbb{L}_{\alpha<\mu} X_\alpha$ such that given any $\alpha < \mu$, X_α is either bounded or $0, 1 \notin X_\alpha$. Let $S = \{\alpha : 0, 1 \notin X_\alpha\}$ and $s = \sup S$.

(a) *If $s = \mu$ then ($*$) holds.*

(b) *If $s < \mu$ then*

(b$_1$) *If $s \notin S$ then*

$$e(X) = \sup_{\beta<s} \left| \prod_{\alpha\leq\beta} X_\alpha \right| \,.\, \sup_{s\leq\alpha<\mu} e(X_\alpha).$$

(b$_2$) *If $s \in S$ then*

$$e(X) = \left| \prod_{\alpha<s} X_\alpha \right| \,.\, \sup_{s\leq\alpha<\mu} e(X_\alpha).$$

Proof. (a) Note that in this case μ is a limit ordinal and one only has to use Theorem 5.15. (b$_1$) Again, one can note that in this case μ is a limit ordinal and the result follows from Theorems 5.15 and 5.9. (b$_2$) The result follows from Corollary 5.12 and Theorem 5.9. □

We finally consider LOTS which have 0 but not 1. Dual statements will hold for the case of LOTS which have 1 but not 0. For this purpose we will be using the following notation. For a LOS X we consider the *left topology* τ_ℓ generated by the subbase $\{\,]a,\rightarrow[\;: a \in X\} \cup \{\,]\leftarrow, a] : a \in X\}$, where $]\leftarrow, a] = \{x \in X : x \leq a\}$. We then denote by $e_\ell(X)$, the extent of (X, τ_ℓ). If Y is a LOTS and $a \in Y$, we denote by Y_a the subset $\{y \in Y : y \leq a\} = \,]\leftarrow, a]$ of Y, which is equal to $[0, a]$ in the case that $0 \in Y$.

Let us start with the following two lemmas, the proofs of which are omitted since they are based on same arguments as those of Lemmas 5.3, 5.4 and 5.5.

Lemma 5.17.

Let X, Y be LOTS, where $0 \in Y$ but $1 \notin Y$. If D is a closed, discrete subset of $X \cdot Y$ then

$$D_X = \big\{x \in X : \text{ there exists } y \in Y \text{ such that } (x, y) \in D\big\}$$

is a closed, discrete subset of (X, τ_ℓ).

Lemma 5.18.

Let X, Y be LOTS, where $0 \in Y$ but $1 \notin Y$. If D is a closed, discrete subset of (X, τ_ℓ), then $D \times \{0\}$ is a closed, discrete subset of $X \cdot Y$. In particular, $e(X \cdot Y) \geq e_\ell(X)$.

Theorem 5.19.

Let X, Y be LOTS, where $0 \in Y$ but $1 \notin Y$.

(a) *If every $x \in X$ has an immediate successor $x^+ \in X$, then $e(X \cdot Y) = e(X) \,.\, \sup_{a \in Y} e(Y_a) = e_\ell(X) \,.\, \sup_{a \in Y} e(Y_a)$.*

(b) *If there exists $x \in X$ that does not have an immediate successor, in particular if $1 \in X$, then $e(X \cdot Y) = e_\ell(X) \,.\, e(Y)$.*

Proof. (a) Let us note that in this case $1 \notin X$. Suppose D is a closed, discrete subset of $X \cdot Y$ and consider D_X as in Lemma 5.17. For $x \in D_X$, let $D_x = \{y \in Y : (x, y) \in D\}$. Now $(x^+, 0)$ has a neighbourhood in $X \cdot Y$ that intersects at most one point of D, consequently there exists $a \in Y$ such that $D_x \subseteq [0, a]$. It follows that $|D| \leq e_\ell(X) \,.\, \sup_{a \in Y} e(Y_a)$. Therefore, $e(X \cdot Y) \leq e_\ell(X) \,.\, \sup_{a \in Y} e(Y_a)$.

Conversely, given a closed, discrete subset D of Y_a, $a \in Y$, and $x \in X$ we have that $\{x\} \times D$ is a closed, discrete subset of $X \cdot Y$. Thus, $e(X \cdot Y) \geq e(Y_a)$ for every $a \in Y$, and $e(X \cdot Y) \geq \sup_{a \in Y} e(Y_a)$ follows. Together with $e(X \cdot Y) \geq e_\ell(X)$ (Lemma 5.18) we then obtain $e(X \cdot Y) \geq e_\ell(X) \,.\, \sup_{a \in Y} e(Y_a)$, and the required equality follows.

(b) Given a closed, discrete subset D of $X \cdot Y$, we have

$$D = \bigcup_{x \in D_X} \big(\{x\} \times D_x\big) \leq e_\ell(X) \,.\, e(Y),$$

due to Lemma 5.17, so that $e(X \cdot Y) \leq e_\ell(X) \,.\, e(Y)$. On the other hand, if $x \in X$ does not have an immediate successor, $\{x\} \times D$ is a closed, discrete subset of $X \cdot Y$ for every closed, discrete subset D of Y. Consequently, $e(X \cdot Y) \geq e(Y)$, and together with Lemma 5.18 we obtain $e(X \cdot Y) \geq e_\ell(X) \,.\, e(Y)$ and equality follows. □

6. Lindelöf number

Let us recall the following cardinal invariant.

Definition 6.1.

For a topological space X, the *Lindelöf number* $\ell(X)$ is the least infinite cardinal number κ such that every open cover of X has an open refinement of cardinality $\leq \kappa$.

Thus, a regular topological space X has the Lindelöf property if and only if $\ell(X) = \aleph_0$. One can also note that $e(X) \leq \ell(X)$ for any topological space X.

Remember that if X is a GO-space and U is a subset of X, then a (pseudo)gap (A, B) is said to be *covered* by U if there is a convex set V such that $V \subseteq U$, $V \cap A \neq \emptyset$ and $V \cap B \neq \emptyset$. In other words, there exist $a \in A$ and $b \in B$ such that $[a, b] \subseteq U$. A cover $\mathcal{U}$ of X is said to *cover* the (pseudo)gap (A, B) if $\mathcal{U}$ has an element which covers (A, B). The following lemma is known (for LOTS see [6]).

Lemma 6.2.

An open cover $\mathcal{U}$ of a GO-space X has a finite subcover if every gap and pseudogap of X is covered by $\mathcal{U}$.

Definition 6.3.

A gap (A, B) of a GO-space X is said to be a *two-sided κ-gap* if A has a cofinal subset of cardinality $\leq \kappa$ and B has a coinitial subset of cardinality $\leq \kappa$.

Definition 6.4.

For a GO-space X, the *gap number* $g(X)$ is the least infinite cardinal number κ such that every gap in X is a two-sided κ-gap and every pseudogap in X is a κ-pseudogap.

Now let $(X, \tau, \leqslant)$ be a GO-space and $\mathcal{U}$ an open cover of X. Denote by $F_{\mathcal{U}}$, the set of all gaps and pseudogaps of X which are not covered by $\mathcal{U}$. It can be easily seen that $F_{\mathcal{U}}$ is closed in X^+ and so is compact. Indeed, for every $U \in \mathcal{U}$, let $U' = U \cup \{c \in X^+ - X : c \text{ is covered by } U\}$. Then U' is open in X^+ and $F_{\mathcal{U}} = X^+ - \bigcup\{U' : U \in \mathcal{U}\}$.

Lemma 6.5.

Let $(X, \tau, \leqslant)$ have gap number $g(X) \leq \kappa$. Let $\mathcal{U}$ be an open cover of X. If $X^+ - F_{\mathcal{U}}$ can be decomposed into $\leq \kappa$ convex components, then $\mathcal{U}$ has a subcover of cardinality $\leq \kappa$.

Proof. Let G_α, $\alpha \leq \kappa$, be the convex components of $X^+ - F_{\mathcal{U}}$. Since $F_{\mathcal{U}}$ is a closed set, the convex components G_α are open in X^+. Let $H_\alpha = G_\alpha \cap X$, then $\{H_\alpha : \alpha \leq \kappa\}$ is a disjoint open cover of X by convex sets. Regard H_α as a GO-space covered by the open cover $\mathcal{U}$. Then $\mathcal{U} \cap H_\alpha$ covers every gap and pseudogap of H_α except possibly its endgaps. Select an arbitrary point a of H_α. If H_α has a maximal point, then by Lemma 6.2, $H'_\alpha = \{x \in H_\alpha : x \geq a\}$ is covered by finitely many elements of $\mathcal{U}$. If $(H_\alpha, \emptyset)$ is an endgap of H_α, then it determines a two-sided κ-gap or κ-pseudogap. In either case there is a cofinal set $(a_\gamma)_\gamma$ in H_α of cardinality $\leq \kappa$, which one can assume to satisfy $a_\gamma \geqslant a$ for all γ. By Lemma 6.2 we conclude that $[a, a_\gamma]$ is covered by finitely many elements of $\mathcal{U}$ for every γ and so H'_α is covered by $\leq \kappa$ many elements of $\mathcal{U}$. We apply the same argument to the left half of H_α to conclude that H_α is covered by $\leq \kappa$ many elements of the open cover $\mathcal{U}$. Consequently, X can be covered by $\leq \kappa$ many elements of the open cover $\mathcal{U}$. $\square$

Proposition 6.6.

The following are equivalent for a GO-space $(X, \tau, \leqslant)$:

(a) *X has Lindelöf number $\ell(X) \leq \kappa$.*

(b) (b_1) *X has gap number $g(X) \leq \kappa$;*

(b_2) *X has extent $e(X) \leq \kappa$.*

(c) (c_1) *X has gap number $g(X) \leq \kappa$;*

(c_2) *for every compact set $F \subset X^+ - X$, $X^+ - F$ is decomposed into $\leq \kappa$ convex components.*

Proof. (a) $\Rightarrow$ (b) As mentioned above, $e(X) \leq \ell(X)$, so that $e(X) \leq \kappa$. The fact that every gap is a two-sided κ-gap and every pseudogap is a κ-pseudogap is not difficult to see. Indeed, if for example (A, B) is a gap, then $\mathcal{U} = \{]\leftarrow, a[: a \in A\} \cup \{B\}$ and $\mathcal{V} = \{A\} \cup \{]b, \rightarrow[: b \in B\}$ are open covers of X, so that (a) implies that (A, B) is a two-sided κ-gap.

(b) $\Rightarrow$ (c) For every compact set $F \subset X^+ - X$, the convex components of $X^+ - F$ give rise to an open disjoint cover of X and so there are $\leq \kappa$ many due to (b$_2$).

(c) $\Rightarrow$ (a) This follows from Lemma 6.5. □

The equivalence of (a) and (b) in Proposition 6.6 for $\kappa = \aleph_0$ was proved in [3] while that of (a) and (c), again for $\kappa = \aleph_0$, was proved in [2]. Let us note that the equivalence of (a) and (b) in Proposition 6.6 gives the following result.

Corollary 6.7.

If $(X, \tau, \leqslant)$ is a GO-space, then $\ell(X) = e(X) . g(X)$.

We now turn to the lexicographic product of LOTS. In [3], Faber notes that the LOTS $]0,1[$ is Lindelöf but $]0,1[\cdot]0,1[$ is not, while on the other hand, $[0,\omega_0[\cdot[0,\omega_1[$ is a Lindelöf LOTS but $[0,\omega_1[$ is not. We now apply the results of Section 5 to study the behaviour of the Lindelöf number in lexicographic products. Let us begin with the following observation.

Lemma 6.8.

Let X_α be LOS for every $\alpha < \mu$ and let $(c_\alpha)_{\alpha<\mu} \in \mathbb{L}_{\alpha<\mu} X_\alpha^+ \setminus \mathbb{L}_{\alpha<\mu} X_\alpha$. Let γ be the smallest ordinal such that $c_\gamma \in X_\gamma^+ \setminus X_\gamma$. Then (A, B), where

$$A = \Big]\leftarrow, (c_\alpha)_{\alpha<\mu}\Big[\cap X \quad \text{and} \quad B = \Big](c_\alpha)_{\alpha<\mu}, \rightarrow\Big[\cap X,$$

gives a gap in $X = \mathbb{L}_{\alpha<\mu} X_\alpha$ if one of the following conditions holds:

(a) *$\gamma = 0$;*

(b) *$\gamma \neq 0$ and c_γ is not an endgap in X_γ;*

(c) *$\gamma \neq 0$, c_γ is the left endgap in X_γ and $(c_\alpha)_{\alpha<\gamma} \in \mathbb{L}_{\alpha<\gamma} X_\alpha$ does not have an immediate predecessor;*

(d) *$\gamma \neq 0$, c_γ is the left endgap in X_γ, $(c_\alpha)_{\alpha<\gamma} \in \mathbb{L}_{\alpha<\gamma} X_\alpha$ has an immediate predecessor and $\mathbb{L}_{\gamma\leq\alpha<\mu} X_\alpha$ has a right endgap;*

(e) *$\gamma \neq 0$, c_γ is the right endgap in X_γ and $(c_\alpha)_{\alpha<\gamma} \in \mathbb{L}_{\alpha<\gamma} X_\alpha$ does not have an immediate successor;*

(f) *$\gamma \neq 0$, c_γ is the right endgap in X_γ, $(c_\alpha)_{\alpha<\gamma} \in \mathbb{L}_{\alpha<\gamma} X_\alpha$ has an immediate successor and $\mathbb{L}_{\gamma\leq\alpha<\mu} X_\alpha$ has a left endgap.*

It is not difficult to see that any gap (A, B) in $X = \mathbb{L}_{\alpha<\mu} X_\alpha$ must have one of the forms listed in (a)–(f) of Lemma 6.8. One can also note that in the case that X_α is bounded for all $\alpha > 0$ then (A, B) always gives a gap and every gap in X is obtained in this way. It also follows that in this particular case, every gap of X is a two-sided κ-gap if and only if every gap of X_α is a two-sided κ-gap for every $\alpha < \mu$, that is

Proposition 6.9.

Let $X = \mathbb{L}_{\alpha<\mu} X_\alpha$, where X_α is a bounded LOTS for all $\alpha > 0$, then $g(X) = \sup\limits_{\alpha<\mu} g(X_\alpha)$.

Consequently, due to Theorem 5.9 and Corollary 6.7, we have

Theorem 6.10.

Let $X = \mathbb{L}_{\alpha<\mu} X_\alpha$, where X_α is a bounded LOTS for all $\alpha > 0$, then $\ell(X) = \sup\limits_{\alpha<\mu} \ell(X_\alpha)$.

If one has a product of the type in Theorem 5.16, one can use that same theorem, together with Lemma 6.8 and Corollary 6.7, to find the Lindelöf number of this product. For convenience we introduce the following notations. For a LOS X and $x \in X$, we denote the cofinality of the set $]\leftarrow, x[$ by $\mathrm{cof}(x)$, the coinitiality of the set $]x, \rightarrow[$ by $\mathrm{coi}(x)$, the cofinality of X by $\mathrm{cof}(X)$ and the coinitiality of X by $\mathrm{coi}(X)$. Also, for a LOS X, let

$$\mathrm{nis}(X) = \big\{x \in X : x \text{ does not have an immediate successor}\big\},$$
$$\mathrm{nip}(X) = \big\{x \in X : x \text{ does not have an immediate predecessor}\big\}.$$

Let us first consider the case of two spaces and therefore, apply Theorem 5.11 together with Lemma 6.8 and Corollary 6.7. We then obtain the following result.

Theorem 6.11.

Suppose X, Y are LOTS such that $0, 1 \notin Y$, then

$$g(X \cdot Y) = g(X) \,.\, g(Y) \,.\, \sup\{\mathrm{cof}(x) : x \in \mathrm{nip}(X)\} \,.\, \sup\{\mathrm{coi}(x) : x \in \mathrm{nis}(X)\}, \qquad \ell(X \cdot Y) = |X| \,.\, \ell(Y).$$

Proof. The first equality follows directly from Lemma 6.8 for two spaces. For the second equality, we have $\ell(X \cdot Y) = e(X \cdot Y) \,.\, g(X \cdot Y) = |X| \,.\, e(Y) \,.\, g(X) \,.\, g(Y) = |X| \,.\, \ell(Y)$. □

For arbitrary products we use Theorems 5.16 and 6.11, Lemma 6.8 and Corollary 6.7 to obtain the following result for the Lindelöf number.

Theorem 6.12.

Consider the product $X = \mathbb{L}_{\alpha<\mu} X_\alpha$ such that given any $\alpha < \mu$, X_α is either bounded or $0, 1 \notin X_\alpha$. Let $S = \{\alpha : 0, 1 \notin X_\alpha\}$ and $s = \sup S$.

(a) *If $s = \mu$ then*

$$\ell(X) = \sup_{\beta<\mu} \left| \prod_{\alpha\leq\beta} X_\alpha \right|.$$

(b) *If $s < \mu$ then*

(b$_1$) *If $s \notin S$ then*

$$\ell(X) \;=\; \sup_{\beta<s} \left| \prod_{\alpha\leq\beta} X_\alpha \right| \,.\, \sup_{s\leq\alpha<\mu} \ell(X_\alpha).$$

(b$_2$) *If $s \in S$ then*

$$\ell(X) \;=\; \left| \prod_{\alpha<s} X_\alpha \right| \,.\, \sup_{s\leq\alpha<\mu} \ell(X_\alpha).$$

Proof. We only need to note that in case (a), $g(X) \leq \sup_{\beta<\mu} \left|\prod_{\alpha\leq\beta} X_\alpha\right|$. In case (b$_1$), $g(X) = \kappa_a \,.\, \sup_{s\leq\alpha<\mu} g(X_\alpha)$, where $\kappa_a \leq \sup_{\beta<s} \left|\prod_{\alpha\leq\beta} X_\alpha\right|$. Finally, in case (b$_2$), $g(X) = \kappa_b \,.\, \sup_{s\leq\alpha<\mu} g(X_\alpha)$, where $\kappa_b \leq \left|\prod_{\alpha<s} X_\alpha\right|$. □

Finally, we can again use Lemma 6.8 to find the Lindelöf number of a product $X \cdot Y$, where $0 \in Y$ but $1 \notin Y$. For the sake of convenience, for a LOS X we define the *internal gap number* $g_i(X)$ of X to be the least infinite cardinal number such that every *interior* gap (i.e. a gap that is not an endgap) of X is a two-sided κ-gap. Using Theorem 5.19 together with Lemma 6.8 and Corollary 6.7, we obtain the following result.

Theorem 6.13.

Suppose X, Y are LOTS such that $0 \in Y$ but $1 \notin Y$, then

(a) *If every $x \in X$ has an immediate successor $x^+ \in X$.*

$$g(X \cdot Y) = g(X) \,.\, g_i(Y), \qquad \ell(X \cdot Y) = \ell(X) \,.\, \sup_{a\in Y} \ell(Y_a).$$

(b) *If there exists $x \in X$ that does not have an immediate successor, in particular if $1 \in X$, then*

$$\begin{aligned} g(X \cdot Y) &= g(X) \,.\, g(Y) \,.\, \sup\{\mathrm{coi}(x) : x \in \mathrm{nis}(X)\}, \\ \ell(X \cdot Y) &= e_\ell(X) \,.\, g(X) \,.\, \sup\{\mathrm{coi}(x) : x \in \mathrm{nis}(X)\} \,.\, \ell(Y). \end{aligned}$$

Proof. (a) The equality for the gap number follows directly from Lemma 6.8 for two spaces. We now have $\ell(X \cdot Y) = e(X \cdot Y) \,.\, g(X \cdot Y) = e(X) \,.\, \sup_{a \in Y} e(Y_a) \,.\, g(X) \,.\, g_i(Y) = \ell(X) \,.\, \sup_{a \in Y} e(Y_a) \,.\, g_i(Y) = \ell(X) \,.\, \sup_{a \in Y} \ell(Y_a)$, where the last equality in (a) follows from the fact that $g_i(Y) = \sup_{a \in Y} g(Y_a)$. Indeed, since every gap of Y_a is an internal gap of Y, it follows that $g(Y_a) \leq g_i(Y)$. From the arbitrariness of $a \in Y$ we then have $\sup_{a \in Y} g(Y_a) \leq g_i(Y)$. Suppose for contradiction that $\kappa = \sup_{a \in Y} g(Y_a) < g_i(Y)$, then there exists an internal gap (A, B) in Y that is not a two-sided κ-gap. Take any $b \in B$, it then follows that $(A, B \cap [0, b])$ is a gap of Y_b that is not a two-sided κ-gap, contradicting $g(Y_b) \leq \kappa$.

(b) Again, from Lemma 6.8 for two spaces, we get that $g(X \cdot Y) = g(X) \,.\, g_i(Y) \,.\, \mathrm{cof}(Y) \,.\, \sup\{\mathrm{coi}(x) : x \in \mathrm{nis}(X)\} = g(X) \,.\, g(Y) \,.\, \sup\{\mathrm{coi}(x) : x \in \mathrm{nis}(X)\}$. Consequently, $\ell(X \cdot Y) = e(X \cdot Y) \,.\, g(X \cdot Y) = e_\ell(X) \,.\, e(Y) \,.\, g(X) \,.\, g(Y) \,.\, \sup\{\mathrm{coi}(x) : x \in \mathrm{nis}(X)\} = e_\ell(X) \,.\, g(X) \,.\, \sup\{\mathrm{coi}(x) : x \in \mathrm{nis}(X)\} \,.\, \ell(Y)$. □

To conclude, we consider the following example.

Example 6.14.
Let $Y = [0, \omega_0[\, + \, ([0, \omega_1[\, \cdot \, \mathbb{Z})$. Let us first note that $0 \in Y$ but $1 \notin Y$. Given any $a \in Y$, the subspace $[0, a] \subseteq Y$ is a countable discrete space and therefore, $e(Y_a) = \aleph_0$. On the other hand, $e(Y) = \aleph_1$.

If one takes $X = [0, \omega_0[$, then $e(X \cdot Y) = e(X) \,.\, \sup_{a \in Y} e(Y_a) = \aleph_0$, so that by Theorem 6.13, $\ell(X \cdot Y) = \aleph_0 \,.\, \aleph_0 . \aleph_0 = \aleph_0$. On the other hand, if one takes $X = [0, 1] \subseteq \mathbb{R}$, then $e(X \cdot Y) = e_\ell(X) \,.\, e(Y) = \aleph_0 \,.\, \aleph_1 = \aleph_1$, so again by Theorem 6.13, $\ell(X \cdot Y) = \aleph_0 \,.\, \aleph_0 \,.\, \aleph_0 \,.\, \aleph_1 = \aleph_1$.

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